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Horizontal Connection and Horizontal Mean Curvature in Carnot Groups

Kang Hai TAN Xiao Ping YANG
Department of Applied Mathematics, Nanjing University of Science and Technology, Nanjing 210094, *P. R. China*
*E-mail: tankanghai*2000*@yahoo.com.cn yangxp@mail.njust.edu.cn*

Abstract In this paper we give a geometric interpretation of the notion of the horizontal mean curvature which is introduced by Danielli–Garofalo–Nhieu and Pauls who recently introduced sub-Riemannian minimal surfaces in Carnot groups. This will be done by introducing a natural nonholonomic connection which is the restriction (projection) of the natural Riemannian connection on the horizontal bundle. For this nonholonomic connection and (intrinsic) regular hypersurfaces we introduce the notions of the horizontal second fundamental form and the horizontal shape operator. It turns out that the horizontal mean curvature is the trace of the horizontal shape operator.

Keywords Carnot groups, Nonholonomic connection, Horizontal mean curvature, Sub-Riemannian minimal surfaces
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1 Introduction and Preliminaries

In [1] Garofalo and Nhieu introduced the sub-Riemannian version of the Plateau problem (sub-Riemannian minimal surface problem). Roughly speaking, this generalization of the classical Plateau problem is based on an appropriate sub-Riemannian version of perimeter generalizing the classical one. The existence of the generalized solutions (H-Caccioppoli sets or H-finite perimeter sets) of the Plateau problem in Carnot–Carathéodory (CC) manifolds was established in [1]. A sub-Riemannian manifold $(M, \Delta, \langle\cdot,\cdot\rangle_c)$ is a smooth manifold M with a distribution Δ (a subbundle of the tangent bundle TM) which is (locally) generated by vector fields $X = \{X_1, \ldots, X_k\}$ and is endowed with a fiberwise inner product $\langle\cdot,\cdot\rangle_c$ (sub-Riemannian metric), ($\langle\cdot,\cdot\rangle_c$ is usually realized as the restriction on Δ of a Riemannian metric $\langle\cdot,\cdot\rangle$ of M), such that $X = \{X_1, \ldots, X_k\}$ is orthonormal. A particularly interesting class of sub-Riemannian manifolds are Carnot groups. We recall that a *Carnot group* G is a connected, simply connected Lie group whose Lie algebra $\mathscr{G}$ admits the grading $\mathscr{G} = V_1 \bigoplus \cdots \bigoplus V_l$, with $[V_1, V_i] = V_{i+1}$, for any $1 \le i \le l-1$ and $[V_1, V_l] = 0$. Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathscr{G}$ with $n = \sum_{i=1}^{l} \dim(V_i)$. Let $X_i(g) = (L_g)_* e_i$ for $i = 1, \ldots, k := \dim(V_1)$ where $(L_g)_*$ is the differential of the left translation $L_g(g') = gg'$ and let $Y_i(g) = (L_g)_* e_{i+k}$ for $i = 1, \ldots, n-k$. We call the system of left-invariant vector fields $HG := V_1 = \operatorname{span}\{X_1, \ldots, X_k\}$ the *horizontal bundle* of G and its *horizontal fiber* at x is denoted by HG_x. We will denote by $C^r(\Omega, HG)$ the set of all C^r smooth sections defined in the open set Ω. We will fix a Riemannian metric $\langle\cdot,\cdot\rangle$ on G such that $\{X_1(g), \ldots, X_k(g), Y_1(g), \ldots, Y_{n-k}(g)\}$ is an orthonormal basis of T_gG for any $g \in G$ (its restriction on HG is a canonical sub-Riemannian metric $\langle\cdot,\cdot\rangle_c$). By the Baker–Hausdorff–Campbell formula we can identify G with $\mathbb{R}^n$ ($\mathscr{G}$) with a group law. We will denote the *homogeneous dimension* of G by $Q := \sum_{i=1}^{l} i \dim(V_i)$.

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Intrinsic regular hypersurfaces and horizontal mean curvature are two important notions in the study of the sub-Riemannian minimal surface problem. About the former, in their recent papers [2–5] Franchi, Serapioni and Serra Cassano studied some basic properties and gave structure theorems for sets of H-finite perimeter in Carnot groups.

Definition 1.1 (Intrinsic regular hypersurface [3]) *Let G be a Carnot group. We say that $S \subset G$ is an intrinsic regular hypersurface if, for every $x \in S$, there exist a neighborhood $\mathscr{U}$ of x and a function $f \in C^1_H(\mathscr{U})$ such that*

$$S \cap \mathscr{U} = \{y \in \mathscr{U} : f(y) = 0\} \text{ and } \nabla_H f(y) \neq 0 \text{ for } y \in \mathscr{U},$$

where $C^r_H(\mathscr{U})$ denotes the set of all continuous functions f in $\mathscr{U}$ such that $X_{i_1}X_{i_2}\cdots X_{i_j}f$ are continuous for any $i_1, \dots, i_j \in \{1, \dots, k\}$ and $j = 1, \dots, r$; $\nabla_H f := (X_1 f, \dots, X_k f)$ is the horizontal gradient of f identified with a section of HG, $\nabla_H f(x) = \sum_{i=1}^k X_i f(x) X_i \in HG_x$. Here $X_i f$ is understood in the distribution sense.

Note that $C^r(\mathscr{U}) \subset C^r_H(\mathscr{U})$, where $C^r(\mathscr{U})$ is the space of Euclidean r-order smooth functions. In general C^r_H functions are not C^r. If an intrinsic regular hypersurface S is C^1 (i.e. the defining functions are C^1), there are no characteristic points in S. Here by a *characteristic point* on a smooth hypersurface S we mean a point x whose tangent space $T_x S$ contains the horizontal fiber HG_x at x. In other words, if $n^E(x)$ is the Euclidean normal vector of S at x, x is a characteristic point if and only if the *horizontal normal* of S at x $n^{\mathscr{H}}(x) := \sum_{i=1}^k \langle n^E(x), X_i(x)\rangle_E X_i(x) \in HG_x$ is vanishing, where $\langle\cdot,\cdot\rangle_E$ is the Euclidean metric, see Remark 3.3 and Lemma 4.1. So all smooth hypersurfaces which do not contain characteristic points are intrinsic regular hypersurfaces. But most smooth hypersurfaces contain a small subset of characteristic points. The smallness means that the set $C(S)$ of all characteristic points of a smooth hypersurface S has $\mathscr{H}^{Q-1}$ (the $Q-1$ dimension Hausdorff measure with respect to the CC metric) or Riemannian surface measure zero (even more smaller if S is smooth enough), see [6]. Conversely, C^1_H hypersurfaces may not possess any differentiability in the usual sense. For technical reasons and simplicity, from the next section we will assume that all intrinsic hypersurfaces are (locally) smooth (C^1 at least).

Franchi, Serapioni and Serra Cassano characterized H-finite perimeter sets in Carnot groups of step two. They showed that the reduced boundary of any H-finite perimeter set is H-rectifiable, i.e., up to a subset of $\mathscr{H}^{Q-1}$ measure zero it is the union of countable compact subsets each of which is contained in an intrinsic regular hypersurface. This result is one of the most exciting results in the developing the geometric measure theory in the setting of sub-Riemannian geometry.

As for horizontal mean curvature of a hypersurface, Pauls [7] and Danielli–Garofalo–Nhien [8] independently suggested two equivalent notions. We recall only the one in [8], see also [9]. Let x be a noncharacteristic point of a C^2 smooth hypersurface S whose defining function is f (i.e. $f \in C^2(G)$, $S = \{x \in G : f(x) = 0\} = \partial\Omega$ where $\Omega = \{x \in G : f(x) < 0\}$ and $\nabla f = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_i} \neq 0$ in S). By continuity there exists a neighborhood $\mathscr{U}$ of x such that $\mathscr{U} \cap S$ is intrinsic regular. Then the unit horizontal normal vector is $\mathscr{V}(y) = \sum_{i=1}^k \mathscr{V}_i(y) X_i$ where $\mathscr{V}_i(y) = \frac{X_i f(y)}{|\nabla_H f(y)|}$ for $y \in \mathscr{U}$.

Definition 1.2 (Horizontal mean curvature [8]) *Let x, S, f and $\mathscr{V}$ be as above. Then the horizontal mean curvature of S at x is defined as*

$$H_X(x) := \frac{1}{Q-1}\sum_{i=1}^k X_i(\mathscr{V}_i(x)) = \frac{1}{Q-1}\sum_{i=1}^k X_i\left\{\frac{X_i f(x)}{|\nabla_H f(x)|}\right\}. \tag{1.1}$$

We first remark that H_X does not depend on the choice of defining functions. This follows from the fact that if f_1, f_2 are two defining functions of S, then there exists a C^1 function h such that $f_1 = hf_2$ (recalling $X_i f = \langle\nabla f, X_i\rangle_E$). Definition 1.2 is just an analogue of the definition of the mean curvature of smooth hypersurfaces in Euclidean spaces $\mathbb{R}^n$, see e.g. [10].

In fact, following the lines of Chapters 1 and 2 of [10], one can develop in Carnot groups an analogue of the calculus in Euclidean domains. However this generalization is only from the viewpoint of analysis. From Definition 1.2, it does not seem to imply any geometry meaning. This note is devoted to giving a geometric interpretation of the horizontal mean curvature. We will introduce in the horizontal bundle HG a nonholonomic connection—*horizontal connection* D, see Section 2 for details. The notion of a nonholonomic connection dates back to Cartan's address at the 1928 International Congress of Mathematicians at Bologna, Italy, see [11], [12] and [13]. Because of the special structure of Carnot groups it turns out that the nonholonomic connection can be uniquely determined by the following formula $D_X Y = \sum_{i=1}^k X\langle Y, X_i\rangle X_i$ for any $X, Y \in C^1(\Omega, HG)$, and it is the restriction and projection on the horizontal bundle HG of the Levi–Civita connection with respect to $\langle\cdot,\cdot\rangle$. With the connection D, we can define the horizontal second fundamental form and horizontal shape operator on smooth intrinsic regular hypersurfaces. We define our horizontal mean curvature as the trace of the horizontal shape operator. We prove that this definition coincides with Definition 1.2.

The paper is organized as follows. In Section 2 we introduce a nonholonomic connection on Carnot groups. Some properties of the nonholonomic connection are developed. The horizontal second fundamental form for smooth intrinsic regular hypersurfaces is defined in Section 3. The theorem which states that the horizontal mean curvature is the trace of the horizontal shape operator is the main result of this paper. In the last section, we give some comments and possible extensions.

2 Horizontal Connection in Carnot Groups

As in the last section, G identified with $\mathbb{R}^n$ is a Carnot group with horizontal bundle $HG = \operatorname{span}\{X_1, \dots, X_k\}$. Fix the inner product $\langle\cdot,\cdot\rangle$ in TG such that the system of left-invariant vector fields $\{X_1, \dots, X_k, Y_1, \dots, Y_{n-k}\}$ is an orthonormal basis of TG. We follow Cartan [11] to give the definition of the nonholonomic connections on G.

Definition 2.1 (Nonholonomic connections) *Let Ω be an open set of G. A nonholonomic connection on HG is an operator*

$$\widetilde{D} : C^1(\Omega, HG) \times C^1(\Omega, HG) \to C(\Omega, HG)$$

satisfying:

1 $\widetilde{D}_X Y$ *is R-linear in both arguments*;

2 $\widetilde{D}_X Y$ *is $C^1(\Omega)$-linear in the argument of X*;

3 *The Leibniz rule holds*; $\widetilde{D}_X(fY) = (Xf)Y + f\widetilde{D}_X Y$,

4 $\widetilde{D}$ *is compatible with respect to $\langle\cdot,\cdot\rangle_c$, that is,*

$$X\langle Y, Z\rangle_c = \langle \widetilde{D}_X Y, Z\rangle_c + \langle Y, \widetilde{D}_X Z\rangle_c \text{ for any } X, Y, Z \in C^1(\Omega, HG). \tag{2.1}$$

It is easily seen that any nonholonomic connection $\widetilde{D}$ can be (uniquely) written as

$$\widetilde{D}_X Y = \sum_{i=1}^k X(Y^i)X_i + \sum_{i,j=1}^k Y^i \widetilde{\omega}_i^j(X)X_j \text{ for any } X, Y = \sum_{i=1}^k Y^i X_i \in C^1(\Omega, HG),$$

with $\widetilde{\omega}_i^j = -\widetilde{\omega}_j^i$, $i, j = 1, \dots, k$. It can be verified in a routine way that $\widetilde{D}_X Y(x)$ depends only on the value of X at x and the evaluation of Y in a neighborhood of x.

There is a natural nonholonomic connection

$$D_X Y = \sum_{i=1}^k X(Y^i)X_i$$

on HG. That is, we choose $\Gamma_{ij}^l := \omega_i^l(X_j) \equiv 0, i, j, l = 1, \dots, k$. We call D the *horizontal connection* on G. We can regard D as the natural generalization of the Euclidean (flat) connection. In fact, if G is commutative, that is, G is Euclidean, then D is just the usual directional derivative in Euclidean spaces.

Theorem 2.2 *D is the restriction and projection on HG of the Levi–Civita connection ∇ with respect to $\langle\cdot,\cdot\rangle$ in the sense that for any $X, Y \in C^1(\Omega, HG)$, D_XY is the projection of ∇_XY on HG.*

Proof Let ∇ be the Levi–Civita connection with respect to the Riemannian metric $\langle\cdot,\cdot\rangle$. From Cozhul's identity

$$\langle\nabla_U V, W\rangle = \frac{1}{2}(U\langle V, W\rangle + V\langle W, U\rangle - W\langle U, V\rangle - \langle V, [U, W]\rangle - \langle W, [V, U]\rangle + \langle U, [W, V]\rangle),$$

for any $U, V, W \in C^2(\Omega, TG)$, we have

$$\langle\nabla_{X_i} X_j, X_l\rangle = \frac{1}{2}(\langle X_i, [X_l, X_j]\rangle - \langle X_j, [X_i, X_l]\rangle - \langle X_l, [X_j, X_i]\rangle) = 0,$$

for $i, j, l = 1, \dots, k$, since $X_1, \dots, X_k, Y_1, \dots, X_{n-k}$ is an orthonormal basis with respect to $\langle\cdot,\cdot\rangle$ and $[X_i, X_j] \in V_2$ for any $i, j = 1, \dots, k$. Thus the connection coefficients for ∇ $\Gamma^l_{ij} = \langle\nabla_{X_i} X_j, X_l\rangle = 0$ when $i, j, l = 1, \dots, k$, and hence the assertion follows.

Corollary 2.3 *D is the unique nonholonomic connection with the symmetry*

$$D_XY = D_YX \text{ for any } X, Y \in HG(\Omega), \tag{2.2}$$

where $HG(\Omega)$ is the set of all left-invariant vector fields with domain Ω.

Proof We first prove that the connection D is symmetric, i.e., (2.2) holds. In fact, by Theorem 2.2 and the symmetry of ∇, i.e., $\nabla_UV - \nabla_VU = [U, V]$ for any $U, V \in C^2(\Omega, TG)$, we have for any $X, Y \in C^1(\Omega, HG)$,

$$D_XY - D_YX = \sum_{i=1}^k \langle\nabla_XY - \nabla_YX, X_i\rangle X_i = \sum_{i=1}^k \langle[X, Y], X_i\rangle X_i. \tag{2.3}$$

Now for $X, Y \in HG(\Omega)$, $D_XY = D_YX$ follows from the fact that $[X, Y] \in V_2$.

Assume $\widetilde{D}$ is a nonholonomic connection on HG such that (2.2) holds. Writing the compatibility equation (2.1) three times with $X, Y, Z \in HG(\Omega)$ cyclicly permuted, we have

$$\begin{aligned} X\langle Y, Z\rangle_c &= \langle\widetilde{D}_XY, Z\rangle_c + \langle Y, \widetilde{D}_XZ\rangle_c, \\ Y\langle Z, X\rangle_c &= \langle\widetilde{D}_YZ, X\rangle_c + \langle Z, \widetilde{D}_YX\rangle_c, \\ Z\langle X, Y\rangle_c &= \langle\widetilde{D}_ZX, Y\rangle_c + \langle X, \widetilde{D}_ZY\rangle_c. \end{aligned}$$

Adding the first and the second equations, then subtracting the third one, using the fact that $D_ZY = D_YZ, D_XY = D_YX$ and $D_ZX = D_XZ$, we get Cozhul's type formula

$$\langle\widetilde{D}_XY, Z\rangle_c = \frac{1}{2}(X\langle Y, Z\rangle_c + Y\langle Z, X\rangle_c - Z\langle X, Y\rangle_c).$$

From the last equation we conclude that $\widetilde{\Gamma}^l_{ij} = \widetilde{\omega}^l_i(X_j) = \langle\widetilde{D}_{X_j} X_i, X_l\rangle_c = 0$ for $i, j, l = 1, \dots, k$. That is, $\widetilde{D} = D$.

Remark 2.4 In general, $D_XY = D_YX$ does not hold for $X, Y \in C^1(\Omega, HG)$. The reason is that $[X, Y]$ may have horizontal components unless X, Y are left-invariant vector fields, see (2.3). Let $[X, Y]^{\mathscr{H}} = \sum_{i=1}^k \langle[X, Y], X_i\rangle X_i$. Then it is easily seen from the proof of Corollary 2.3 that D is the unique nonholonomic connection such that

$$D_XY - D_YX = [X, Y]^{\mathscr{H}}, \tag{2.4}$$

for any $X, Y \in C^1(\Omega, HG)$. The observation of equation (2.4) is fundamental in the proof of the symmetry of the horizontal second fundamental form. In general, for any nonholonomic connection $\widetilde{D}$, it is natural to define on the horizontal bundle HG a torsion tensor

$$T(X, Y) = \widetilde{D}_XY - \widetilde{D}_YX - [X, Y]^{\mathscr{H}},$$

and a curvature tensor

$$C(X, Y) = \widetilde{D}_X\widetilde{D}_Y - \widetilde{D}_Y\widetilde{D}_X - \widetilde{D}_{[X,Y]^{\mathscr{H}}},$$

for any $X, Y \in C^1(\Omega, HG)$. However these notions do not seem to be useful for our purpose. We will not explore them further.

We recall that the horizontal divergence for $X = \sum_{i=1}^k X^i X_i \in C^1(\Omega, HG)$ is

$$\mathrm{div}_H X = \sum_{i=1}^k X_i(X^i).$$

It happens that $\mathrm{div}_H X$ can be defined using the horizontal connection in the usual way:

$$\mathrm{div}_H X = \mathrm{trace}_{\langle\cdot,\cdot\rangle_c}(Y \to D_Y X) := \sum_{i=1}^k \langle X_i, D_{X_i} X\rangle. \tag{2.5}$$

From (2.5) and Theorem 2.2 we have immediately:

Corollary 2.5 *If $X \in C^1(\Omega, HG)$, then $\mathrm{div}_H X = \mathrm{div} X$, where $\mathrm{div} X$ is the usual divergence computed with respect to the Levi–Civita connection ∇ of $\langle\cdot,\cdot\rangle$.*

3 Horizontal Mean Curvature of Intrinsic Regular Hypersurfaces: Another Definition

In this section we follow the steps of procedure in Riemannian geometry (see e.g. [14]) of defining mean curvatures on submanifolds to give a geometric definition of horizontal mean curvatures on intrinsic regular hypersurfaces. In this process we should temporarily forget some concepts in Riemannian geometry such as the usual tangent plane, normal vector, and observe any horizontal (sub-Riemannian) object as standing in the horizontal bundle. In this sense the horizontal connection D provides us with a natural bridge: we can differentiate horizontal vector fields along horizontal vectors, and hence we can define the horizontal second fundamental form $\mathrm{II}(X, Y)$ on intrinsic regular hypersurfaces. The concept of horizontal tangent planes (bundles) (see Definition 3.1) is of paramount importance for our purpose. The fact that the horizontal tangent bundle of a smooth intrinsic regular hypersurface is the projection onto the horizontal bundle of its Riemannian tangent bundle, together with the symmetry of the horizontal connection D (see Equation (2.4)) implies that $\mathrm{II}(X, Y)$ is symmetric. Thus we can define the horizontal mean curvature as the trace of the horizontal shape operator associated with II.

We first give the definition of the horizontal tangent plane of a smooth intrinsic regular hypersurface, which plays a pervasive role in this section.

Definition 3.1 (Horizontal tangent plane) *Let S be a C^2 intrinsic regular hypersurface. Then any $x \in S$ is not characteristic and hence there exists a horizontal normal (see p. 2) $n^{\mathscr{H}}(x)$ at x ($n^{\mathscr{H}}(x)$ is obviously C^1 smooth). We define the horizontal tangent plane $T_x^{\mathscr{H}} S$ at $x \in S$ as*

$$T_x^{\mathscr{H}} S = \{v \in HG_x : \langle v, n^{\mathscr{H}}(x)\rangle_c = 0\}$$

and call

$$T^{\mathscr{H}} S := \bigcup_{x \in S} T_x^{\mathscr{H}} S$$

the horizontal tangent bundle on S.

Since HG_x is a k-dimensional space, $T_x^{\mathscr{H}} S$ is a $(k-1)$-dimensional subspace of HG_x. That is,

$$HG_x = T_x^{\mathscr{H}} S \bigoplus n^{\mathscr{H}}(x). \tag{3.1}$$

By the following proposition, $T_x^{\mathscr{H}} S$ has a significant geometric meaning.

Proposition 3.2 (Blow-up theorem [4]) *Let G be a Carnot group of step 2 and S be a C^1 smooth intrinsic regular hypersurface such that $S \subset \partial E$ for a C^1 smooth, open set E in G. For $x \in S$ and $r > 0$, setting $E_{r,x} := \{y \in G : x.\delta_r(y) \in E\} = \delta_{\frac{1}{r}} L_{x^{-1}} E$ and $T^+(x) := \{y \in$*

$G : \langle \pi_x y, n^{\mathscr{H}}(x)\rangle_c \geq 0\}$, $T(x) := \{y \in G : \langle \pi_x y, n^{\mathscr{H}}(x)\rangle_c = 0\}$, where $\pi_x y = \sum_{i=1}^k y_i X_i(x)$ for $y = (y_1, \ldots, y_k, \ldots, y_n)$, we have

$$\lim_{r\to 0} \chi_{E_{r,x}} = \chi_{T^+} \quad in \quad L^1_{\mathrm{loc}}(G),$$

where χ_E denotes the characteristic function of the set E.

Proposition 3.2 was proven in [4] under a more weaker condition. $T(x)$ is regarded as a "tangent space" in an intrinsic way and is called the *tangent group* of S at x, see [4] (where $L_x T(x)$ is called the *tangent plane* through x). Loosely speaking, $T_x^{\mathscr{H}} S$ is the projection on the horizontal bundle of the Lie algebra of the tangent group.

In the following, we always assume that S is a smooth (at least C^2) intrinsic regular hypersurface and let $\Omega \subset G$ be a smooth open set such that $S \subset \Omega$.

Remark 3.3 It is well known that the horizontal normal vector field $n^{\mathscr{H}}(x)$ can be realized as the projection onto the horizontal bundle of the unit Riemannian normal (also of the unit Euclidean normal), that is, there exists a nonzero function $f(x)$ on S such that

$$n^{\mathscr{H}}(x) := \sum_{i=1}^k \langle n^E(x), X_i(x)\rangle_E X_i(x) = f(x) \sum_{i=1}^k \langle n^{\mathscr{R}}(x), X_i(x)\rangle X_i(x) \quad \text{for } x \in S, \tag{3.2}$$

where $n^{\mathscr{R}}(x)$ denotes the unit Riemannian normal vector field of S with respect to the inner product $\langle\cdot,\cdot\rangle$ and $n^E(x)$ denotes the unit Euclidean normal vector and $\langle\cdot,\cdot\rangle_E$ is the standard Euclidean inner product. Here we identify G with $\mathbb{R}^n$; $n^E(x)$ is the usual normal vector if we regard $S \subset \mathbb{R}^n$, and $X_i(x) = \sum_{i=1}^n a_i^j(x) \frac{\partial}{\partial x_j}$ is regarded as a vector $(a_i^1(x), \ldots, a_i^n(x)) \in \mathbb{R}^n$. The first identity of (3.2) can be regarded as the definition of the horizontal normal vector, see [15] and [16]; the second identity was cited by many authors, see e.g. [7], but we can not find its rigorous proof in the literature. For completeness we give the proof in Section 4. It is natural to ask whether $T_x^{\mathscr{H}} S$ is the projection (with respect to $\langle\cdot,\cdot\rangle$ or $\langle\cdot,\cdot\rangle_E$) onto the horizontal space HG_x of the tangent space of S at x. The answer is yes. In fact, let $T_x S$ be the tangent space of S at x and let $\mathscr{P}T_x S$ denote the projection on HG_x of $T_x S$, that is, $v \in HG_x$ belongs to $\mathscr{P}T_x S$ if and only if there exists a $\widetilde{v} \in T_x S$ such that $v = \sum_{i=1}^k \langle \widetilde{v}, X_i(x)\rangle X_i(x)$. Since $T_x S$ does not contain HG_x, the dimension of $\mathscr{P}T_x S$ is at most $k-1$. On the other hand, if $v \in T_x^{\mathscr{H}} S$, then from (3.2) we have

$$\langle v, f(x)^{-1} n^{\mathscr{R}}(x)\rangle = \langle v, n^{\mathscr{H}}(x)\rangle + \langle v, f(x)^{-1} n^{\mathscr{R}}(x) - n^{\mathscr{H}}(x)\rangle = 0 + 0 = 0,$$

and hence $T_x^{\mathscr{H}} S \subset T_x S$. Thus $T_x^{\mathscr{H}} S \subset \mathscr{P}T_x S$, and hence $T_x^{\mathscr{H}} S = \mathscr{P}T_x S$. The case with respect to the Euclidean metric follows from the same argument.

The above discussion in particular implies the following proposition, which will be used in the proof of the symmetry of the horizontal second fundamental form:

Proposition 3.4 *Let $X, Y \in C^2(S, T^{\mathscr{H}} S)$. Then $[X, Y]^{\mathscr{H}} \in C^1(S, T^{\mathscr{H}} S)$.*

Proof Recalling the definition of $[X, Y]^{\mathscr{H}}$, the statement follows from $[X, Y] \in C^1(S, TS)$ and $T^{\mathscr{H}} S = \mathscr{P}TS$ since $X, Y \in C^2(S, TS)$ as Remark 3.3 showed.

Let $x \in S$ and let $\mathscr{V}(x)$ be the unit horizontal normal vector, that is,

$$\mathscr{V}(x) := \frac{n^{\mathscr{H}}(x)}{|n^{\mathscr{H}}(x)|},$$

where $|n^{\mathscr{H}}(x)|$ is, of course, computed with respect to $\langle\cdot,\cdot\rangle_c$. Since S is an embedded C^2 submanifold of G, it is clear that any vector v in $T_p^{\mathscr{H}} S$ ($p \in S$) can be extended to a vector field in $T^{\mathscr{H}} S$ by first extending v to a vector field V in TS and then projecting V to $T^{\mathscr{H}} S$, and any vector field V in $T^{\mathscr{H}} S$ can be extended to a horizontal vector field in Δ by first extending V to a vector field $\overline{V}$ in TG and then projecting $\overline{V}$ to Δ. If X, Y are vector fields in $T^{\mathscr{H}} S$, we can extend them to horizontal vector fields $\overline{X}, \overline{Y} \in C^1(\Omega, HG)$ on Ω, apply the ambient derivative operator D, and then decompose at points of S to get

$$D_{\overline{X}} \overline{Y}(x) = (D_{\overline{X}} \overline{Y})^\top(x) + (D_{\overline{X}} \overline{Y})^\perp(x), \quad x \in S, \tag{3.3}$$

where $(D_{\overline{X}}\overline{Y})^\top(x)$, $(D_{\overline{X}}\overline{Y})^\perp(x)$ are the projections of $D_{\overline{X}}\overline{Y}(x)$ onto $T^{\mathscr{H}}_x S$ and the direction of $\mathscr{V}(x)$ respectively, see (3.1).

Definition 3.5 (The horizontal second fundamental form) *Let X, Y be vector fields in $T^{\mathscr{H}}S$. We define $\mathrm{II}(X,Y) := (D_{\overline{X}}\overline{Y})^\perp$, where $\overline{X}, \overline{Y}$ are the arbitrarily extended horizontal vector fields of X, Y, respectively, as the horizontal second fundamental form of S.*

Theorem 3.6 *The horizontal second fundamental form $\mathrm{II}(X,Y)$ is:*

1 *Independent of the extension of X and Y;*
2 *Bilinear over $C^1(S)$; and,*
3 *Symmetric in X and Y.*

Proof We first show that the symmetry of II follows from the symmetry of D, (2.4), and Proposition 3.4. Let $\overline{X}, \overline{Y}$ be the arbitrarily extended horizontal vector fields to Ω. Then by (2.4) we have

$$\mathrm{II}(X,Y) - \mathrm{II}(Y,X) = (D_{\overline{X}}\overline{Y} - D_{\overline{Y}}\overline{X})^\perp = ([\overline{X}, \overline{Y}]^{\mathscr{H}})^\perp.$$

Now by Proposition 3.4, $[\overline{X}, \overline{Y}]^{\mathscr{H}}|_S$ belongs to $T^{\mathscr{H}}S$. Therefore $([\overline{X}, \overline{Y}]^{\mathscr{H}})^\perp|_S = 0$, so II is symmetric.

Because $D_X Y(x)$ depends only on $X(x)$, it is clear that $\mathrm{II}(X,Y)$ is independent of the extension chosen for X, and that $\mathrm{II}(X,Y)$ is linear over $C^1(S)$ in X. By symmetry, the same is true for Y.

For the tangent term of $(D_{\overline{X}}\overline{Y})^\top$ in (3.3), we can informally define the *tangent horizontal connection* on S, $D^\top : C^1(S, T^{\mathscr{H}}S) \to C(S, T^{\mathscr{H}}S)$ by $D^\top_X Y := (D_{\overline{X}}\overline{Y})^\top$. The independence of the choice of the extensions of X and Y follows from the same argument as in the proof of Theorem 3.6. Note that we have the symmetry

$$D^\top_X Y - D^\top_Y X = [X,Y]^{\mathscr{H}} \tag{3.4}$$

also by (2.4) and Proposition 3.4. Let $X, Y, Z \in C^1(S, T^{\mathscr{H}}S)$ be arbitrarily extended to Ω. Using the compatibility of D with respect to $\langle\cdot,\cdot\rangle_c$, and evaluating at points of S, we get

$$\begin{aligned} X\langle Y, Z\rangle_c &= \langle D_X Y, Z\rangle_c + \langle Y, D_X Z\rangle_c = \langle (D_X Y)^\top, Z\rangle_c + \langle Y, (D_X Z)^\top\rangle_c \\ &= \langle D^\top_X Y, Z\rangle_c + \langle Y, D^\top_X Z\rangle_c. \end{aligned} \tag{3.5}$$

Therefore we have:

Proposition 3.7 *$D^\top$ is a restricted nonholonomic connection on $T^{\mathscr{H}}S$ endowed with the restricted metric of $\langle\cdot,\cdot\rangle_c$ in the sense that, for any C^1 sections X, Y, Z of $T^{\mathscr{H}}S$, we have:*

1 *$D^\top_X Y$ is R-linear in both arguments;*
2 *$D^\top_X Y$ is $C^1(S)$-linear in the argument of X;*
3 *The Leibniz rule holds: $D^\top_X(fY) = (Xf)Y + f\widetilde{D}_X Y$ for $f \in C^1(S)$;*
4 *(3.5) and (3.4) hold.*

Proposition 3.8 (The Weingarten type equation) *Let $X, Y \in C^1(S, T^{\mathscr{H}}S)$ and N be the horizontal normal vector field. When X, Y, N are arbitrarily extended to Ω, the following equation holds at points of S: $\langle D_X N, Y\rangle = -\langle N, II(X,Y)\rangle$.*

Proof Since $\langle N, Y\rangle_c$ vanishes identically along S, the following holds along S:

$$\begin{aligned} 0 &= X\langle N, Y\rangle = \langle D_X N, Y\rangle + \langle N, D_X Y\rangle \\ &= \langle D_X N, Y\rangle + \langle N, D^\top_X Y + \mathrm{II}(X,Y)\rangle = \langle D_X N, Y\rangle + \langle N, \mathrm{II}(X,Y)\rangle. \end{aligned}$$

Definition 3.9 *The scalar horizontal second fundamental form h is the symmetric bilinear function on $T^{\mathscr{H}}S$ defined by $h(X,Y) = \langle \mathrm{II}(X,Y), \mathscr{V}\rangle_c$. That is, $\mathrm{II}(X,Y) = h(X,Y)\mathscr{V}$. Recall that $\mathscr{V}$ is the unit horizontal normal vector field. From the Riesz representation theorem h uniquely determines an endomorphism of $T^{\mathscr{H}}S$, A, that is, $\langle AX, Y\rangle_c = h(X,Y)$, for all $X, Y \in C^1(S, T^{\mathscr{H}}S)$. We call A the horizontal shape operator of S.*

From the symmetry of II, A is self-adjoint, that is, $\langle AX, Y\rangle_c = \langle X, AY\rangle_c$ for all $X, Y \in C^1(S, T^{\mathscr{H}}S)$. Another way to define A is to let $AX = (D_{\overline{X}}\overline{\mathscr{V}})^\top$, evaluated at points of S, where $\overline{X}$(resp. $\overline{\mathscr{V}}$) is an extension of X(resp. $\mathscr{V}$). Since, for any $f \in C^1(\Omega)$, $D_{\overline{X}}(f\overline{\mathscr{V}}) = fD_{\overline{X}}\overline{\mathscr{V}} + \overline{X}(f)\overline{\mathscr{V}}$, we have $(D_{\overline{X}}(f\overline{\mathscr{V}}))^\top = f(D_{\overline{X}}\overline{\mathscr{V}})^\top$. The last formula in particular implies that AX is independent of the choice of the extensions of X and $\mathscr{V}$. This discussion indirectly proves that $\mathrm{II}(X,Y)$ is independent of the choice of the extensions of X and Y which is given in Theorem 3.6.

For any $x \in S$, A gives a symmetric linear map $A_x : T_x^{\mathscr{H}}S \to T_x^{\mathscr{H}}S$. Then by the symmetry of A_x, A_x has $k-1$ real eigenvalues (recalling $T_x^{\mathscr{H}}S$ is $(k-1)$-dimensional).

Definition 3.10 (Horizontal mean curvature) *The $k-1$ eigenvalues of A_x, $\kappa_1, \dots, \kappa_{k-1}$ are called the horizontal principal curvatures at x and the corresponding eigenspaces are called horizontal principal directions. We define the horizontal mean curvature $\overline{H}_X(x)$ at x as $\frac{1}{Q-1}$ times the trace of A_x, that is, $\overline{H}_X(x) = \frac{1}{Q-1}\sum_{i=1}^{k-1}\kappa_i$, and call the product of $\kappa_1, \dots, \kappa_{k-1}$ the horizontal Gaussian curvature at x.*

Theorem 3.11 *The definition of the horizontal mean curvature given in Definition* 3.10 *coincides with that in Definition* 1.2, *that is,*

$$\overline{H}_X(x) = \frac{1}{Q-1}\sum_{i=1}^{k} X_i(\mathscr{V}^i) \quad \textit{for any } x \in S,$$

where $\mathscr{V} = \sum_{i=1}^{k}\mathscr{V}^i X_i$ is the unit horizontal normal vector field of S.

Proof There exists an orthonormal basis (with respect to the induced metric of $\langle\cdot,\cdot\rangle_c$) of $T^{\mathscr{H}}S$, $\{\tau_1, \dots, \tau_{k-1}\}$, such that $A\tau_i = \kappa_i\tau_i$ for $i = 1, \dots, k-1$ and $\{\tau_1, \dots, \tau_{k-1}, \mathscr{V}\}$ is an orthonormal basis of $HG|_S$. Let $\tau_i = \sum_{j=1}^{k} t_i^j X_j$. Then we have

$$\begin{cases} \mathscr{V}^j\mathscr{V}^l + \displaystyle\sum_{i=1}^{k-1} t_i^j t_i^l = 0, & \text{if } j \neq l; \\ (\mathscr{V}^j)^2 + \displaystyle\sum_{i=1}^{k-1}(t_i^j)^2 = 1 \end{cases} \qquad \text{for } j, l = 1, \dots, k. \tag{3.6}$$

Extending arbitrarily τ_i, $i = 1, \dots, k-1$, and $\mathscr{V}$ to Ω and then evaluating at points of S, we have

$$\begin{aligned}
(Q-1)\overline{H}_X &= \sum_{i=1}^{k-1}\kappa_i = \sum_{i=1}^{k-1}\langle A\tau_i, \tau_i\rangle_c = \sum_{i=1}^{k-1}\langle (D_{\tau_i}\mathscr{V})^\top, \tau_i\rangle_c = \sum_{i=1}^{k-1}\langle D_{\tau_i}\mathscr{V}, \tau_i\rangle_c \\
&= \sum_{i=1}^{k-1}\sum_{j=1}^{k}\langle \tau_i(\mathscr{V}^j)X_j, \tau_i\rangle_c = \sum_{i=1}^{k-1}\sum_{j=1}^{k}\sum_{l=1}^{k}\sum_{r=1}^{k} t_i^l t_i^r X_l(\mathscr{V}^j)\delta_j^r \\
&= \sum_{i=1}^{k-1}\sum_{j=1}^{k}\sum_{l=1}^{k} X_l(\mathscr{V}^j)t_i^j t_i^l = \sum_{j=1}^{k}\sum_{j\neq l=1}^{k}\sum_{i=1}^{k-1} X_l(\mathscr{V}^j)t_i^j t_i^l + \sum_{j=1}^{k}\sum_{i=1}^{k-1} X_j(\mathscr{V}^j)(t_i^j)^2 \\
&= -\sum_{j=1}^{k}\sum_{j\neq l=1}^{k} X_l(\mathscr{V}^j)\mathscr{V}^j\mathscr{V}^l + \sum_{j=1}^{k}\sum_{i=1}^{k-1} X_j(\mathscr{V}^j)(t_i^j)^2 && (3.7) \\
&= \sum_{l=1}^{k} X_l(\mathscr{V}^l)(\mathscr{V}^l)^2 + \sum_{j=1}^{k}\sum_{i=1}^{k-1} X_j(\mathscr{V}^j)(t_i^j)^2 && (3.8) \\
&= \sum_{j=1}^{k} X_j(\mathscr{V}^j)\Big((\mathscr{V}^j)^2 + \sum_{i=1}^{k-1}(t_i^j)^2\Big) = \sum_{j=1}^{k} X_j(\mathscr{V}^j), && (3.9)
\end{aligned}$$

where we have used the first formula of (3.6) to deduce (3.7), used the fact that $\sum_{j=1}^{k}(\mathscr{V}^j)^2 = 1$ to get (3.8) and used the second formula of (3.6) to obtain (3.9).

4 Final Comments

From the above discussion, we have seen that in the setting of sub-Riemannian geometry, at least for Carnot groups, the horizontal connection can be used to define the horizontal mean curvature of intrinsic regular hypersurfaces. It is interesting that such a horizontal connection can be realized as the projection of the Riemannian connection. We point out that restricted connections (of the Riemannian connection) on subbundles of the tangent bundle have been studied by many authors, see [13], [17–19], but have not been used by them to study the geometry of hypersurfaces.

Note that in Section 3 we have used a few special structures of Carnot groups. In fact, with more or less modifications, our arguments can also be applied to more general sub-Riemannian manifolds. More precisely, if we define the horizontal connection on the horizontal bundle as the projection of the Riemannian connection, then such a horizontal connection is a nonholonomic connection in the sense of Definition 2.1 and also satisfies the symmetry of (2.4). Of course the horizontal connection may not have the simple form as in Carnot groups, due to losing the stratified structure. It should be pointed out that the key to proving the symmetry of the horizontal second fundamental form is to prove Proposition 3.4 which can be deduced from (3.2). Fortunately the second identity of (3.2) is valid in the general case.

Lemma 4.1 *Let $\Delta = \operatorname{span}\{X_1, \dots, X_k\}$ be a smooth distribution in $\Omega \subset \mathbb{R}^n$ with $X_i = \sum_{i=1}^n c_i^j \frac{\partial}{\partial x_j}$. Assume Δ satisfies the Hörmander condition and denote by E the tangent bundle generated by Δ. Suppose Δ is endowed with a sub-Riemannian metric $\langle\cdot,\cdot\rangle_c$ such that $\langle\cdot,\cdot\rangle_c$ is the restriction to Δ of a Riemannian metric $\langle\cdot,\cdot\rangle$ on E and $\{X_1, \dots, X_k\}$ is an orthonormal basis of Δ. Let S be a smooth noncharacteristic hypersurface in Ω and*

$$n^{\mathscr{H}}(x) := \sum_{i=1}^{k}\sum_{j=1}^{n} c_i^j(x) n^j(x) X_i \text{ for } x \in S$$

be the horizontal normal vector, where $n^E(x) = \sum_{i=1}^n n^i \frac{\partial}{\partial x_i}$ is the Euclidean normal vector. Then there exists a nonzero function $f(x)$ on S such that

$$n^{\mathscr{H}}(x) = f(x)\sum_{i=1}^{k}\langle n^{\mathscr{R}}(x), X_i\rangle X_i \text{ for } x \in S,$$

where $n^{\mathscr{R}}(x)$ is the Riemannian normal computed with respect to $\langle\cdot,\cdot\rangle$.

Proof We can assume $\{X_1, \dots, X_k, \dots, X_n\}$ is an orthonormal basis of E with respect to $\langle\cdot,\cdot\rangle$. For $i = 1, \dots, n$, let $X_i = \sum_{j=1}^n c_i^j \frac{\partial}{\partial x_j}$; then from the assumption the matrix $C = (c_i^j)_{n\times n}$ is non-singular at points in Ω. Assume $n^{\mathscr{R}}(x) = \sum_{i=1}^n \widetilde{n}^i(x) X_i(x)$ for $x \in S$. Note that the tangent space T_xS is independent of any choice of metrics. Therefore

$$T_xS = \Big\{v = \sum_{i=1}^n v^i \frac{\partial}{\partial x_i} : \langle v, n^E(x)\rangle_E = 0\Big\} = \Big\{u = \sum_{i=1}^n u^i X_i : \langle u, n^{\mathscr{R}}(x)\rangle = 0\Big\}, \tag{4.1}$$

where $\langle\cdot,\cdot\rangle_E$ denotes the Euclidean metric. Let $u = \sum_{i=1}^n u^i X_i = \sum_{j=1}^n (\sum_{i=1}^n c_i^j u^i)\frac{\partial}{\partial x_j}$ be any element in T_xS. From (4.1) we have

$$\sum_{i=1}^n u^i \Big(\sum_{j=1}^n c_i^j n^j(x)\Big) = 0, \tag{4.2}$$

$$\sum_{i=1}^n u^i \widetilde{n}^i(x) = 0. \tag{4.3}$$

Since (4.2) and (4.3) should determine the same solution, we have

$$\sum_{j=1}^n c_i^j n^j(x) = f(x)\widetilde{n}^i(x) \text{ for } i = 1, \dots, n,$$

for some nonzero function $f(x)$. In particular, we have

$$n^{\mathscr{H}}(x) = \sum_{i=1}^{k} \widetilde{n}^i(x) X_i = f(x) \sum_{i=1}^{k} \langle n^{\mathscr{R}}(x), X_i \rangle X_i.$$

By Lemma 4.1, when we try to extend our arguments to sub-Riemannian manifolds, which can not be covered by only one chart, it is natural to define the horizontal normal as the projection of the Riemannian normal onto the horizontal bundle. In this way, the extension can be smoothly carried out with the same results except Theorem 3.11.

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